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Deformable, rigid, and inviscid elliptical inclusions in a homogeneous incompressible anisotropic viscous fluid

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ABSTRACT

The solution for stress, rate of deformation, and vorticity in an incompressible anisotropic viscous cylindrical inclusion with elliptical cross-section embedded in an incompressible, homogeneous anisotropic viscous medium subjected to a far-field homogeneous rate of deformation is presented. The rate of rotation of a single rigid elliptical inclusion is independent of the ratio of the principal viscosity in "foliation-parallel" shortening or extension to that in foliation-parallel shear, $m = \eta_n/\eta_s$, and is hence given by the well-known result for the isotropic medium. An analytical expression shows that a thin, very weak elliptical inclusion rotates as though it were a material line in a homogeneous medium [Kocher, T., Mancktelow, N.S., 2005. Dynamic reverse modeling of flanking structures: a source of quantitative kinematic information. Journal of Structural Geology 27, 1346–1354; Kocher, T., Mancktelow, N.S., 2006. Flanking structure development in anisotropic viscous rock. Journal of Structural Geology 28, 1139–1145]. The sense of slip and slip rate across such an inclusion depends on m. The behavior of an isotropic inclusion with viscosity η^*in a medium deforming in simple shear parallel to its foliation plane, depends on m and $R = \eta^*/\eta_n$; R is the quantity of the same name in Bilby and Kolbuszewski [Bilby, B.A., Kolbuszewski, M.L., 1977. The finite deformation of an inhomogeneity in two-dimensional slow viscous incompressible flow. Proceedings of the Royal Society of London Series A – Mathematical and Physical Sciences 355, 335–353] when the host is isotropic, $m = 1$. R and m determine ranges of qualitatively different behavior in a finite shearing deformation. For $mR = \eta^*/\eta_s < 2$, all inclusions, irrespective of initial aspect ratio and orientation, are stretched to indefinitely large values and their long axis approaches the shear plane. For $mR > 2$, depending on initial aspect ratio, a/b, and orientation to the shear plane, ϕ, the inclusions may either undergo periodic motion or asymptotically approach the shear plane as $a/b \to \infty$. In the former case, a stationary point in ϕ, a/b – phase space occurs at $\phi = 0$ and $(a/b)_C = (\sqrt{m}[1 + \sqrt{R(mR-2)+1}])/(mR-2)$. Initial values in the rather broad vicinity of this point undergo periodic motion. For $R > R_1$, where $m^{0.8}R_1 = [(\eta^*)^5/\eta_n\eta_s^4]^{1/5} \cong 3.40$, by fit to numerically determined values, all initial pairs of ϕ and a/b lead to periodic motion, which may either be a full rotation about the shear plane or an oscillation.

1. Introduction

Inclusions isolated or dispersed in a homogeneous rock volume, may be used to estimate strain, kinematics of flow, stress magnitude, or rheological parameters (Simpson and Schmid, 1983; Lisle et al., 1983; Kanagawa, 1993; Mancktelow et al., 2002; Schmid and Podlachikov, 2003; Kocher and Mancktelow, 2005; Treagus and Lan, 2000). Deformable inclusions have been used in models of clasts in conglomerates (Treagus and Treagus, 2001; Treagus, 2002; Fletcher, 2004). Rigid inclusions have been used to model phenocryst behavior, especially in shear zones (Schmid and Podlachikov, 2003). The behavior of, and deformation around, thin, very weak inclusions has been used to model flanking structures (Grasemann et al., 2003; Exner et al., 2004; Kocher and Mancktelow, 2005, 2006). Since these and similar objects often occur in rock whose well-developed fabrics suggest rheological anisotropy, the behavior of such inclusions in an anisotropic medium is of interest. The solution might also be applied to large-scale inclusions in shear zones (Bellot, 2008) or to continental-scale inhomogeneity (Tommasi and Vauchez, 1997).

Here, I correct typographical errors and simplify the solution from Fletcher (2004) for a deformable elliptical inclusion embedded in a homogeneous anisotropic medium undergoing

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homogeneous far-field deformation. This solution requires that the host and inclusion be homogeneous. When foliation is deflected around the inclusion as in a finite deformation, it provides only an approximation. The homogeneous components of stress, rate of deformation, and vorticity in the inclusion are given, and are used to illustrate inclusion behavior for rotation of a single rigid inclusion in an anisotropic medium, for an inviscid inclusion, and for a deformable isotropic inclusion in foliation-parallel simple shear. The last extends results of Bilby and Kolbuszewski (1977) and Mulchrone and Walsh (2006) for the isotropic matrix.

Solutions for the anisotropic elastic host have been given in the engineering mechanics literature (e.g., Hufenbach and Zhou, 2001; Bhargava and Saxena, 1975; Gao, 1992; Podil'chik, 1997; Rahman, 2002; Ru, 2003). These are not readily compared with the present solution for an anisotropic, incompressible viscous host. Mandal et al. (2005) obtain a result for the infinitesimal rotation of a rigid elliptical inclusion in an anisotropic compressible elastic medium.

2. Amended solution from Fletcher (2004)

The solution for the homogeneous stress components and vorticity in a homogeneous anisotropic elliptical inclusion embedded in a homogeneous anisotropic, incompressible viscous fluid is given in the appendix of Fletcher (2004). This solution is derived from the elastic solution for an incompressible elastic host. Rate of deformation and velocity in the viscous problem are replaced by strain and displacement in the elastic problem. Incompressibility results in symmetry that markedly simplifies the development. A lucid development of the solution for an elliptical cavity in an elastic medium is given by Lekhnitskii (1963, Chapter 3). The inclusion solution is then obtained using the method of Eshelby (1957), and the combined treatment for an application involving a compressible elastic host and inclusion was derived by Fletcher (1968).

Lekhnitskii's expressions for the plane-strain elastic compliances, β_{ij}, are used in the present viscous problem to facilitate comparison with his treatment. To illustrate, the expression for a normal component of strain for an isotropic elastic material undergoing plane deformation is:

$$\varepsilon_{xx} = \frac{(1+\nu_P)}{E}\sigma_{xx} - \frac{\nu_P(1+\nu_P)}{E}\sigma_{yy} = \beta_{11}^{(el)}\sigma_{xx} + \beta_{12}^{(el)}\sigma_{yy} \quad (1)$$

where E is Young's Modulus, ν_P is Poisson's Ratio, and (el) is used to denote an elastic compliance. For an anisotropic viscous fluid, relations of form (1) are written between the rate of deformation and stress components, and coefficients β_{ij} are used. Rate of deformation components for plane flow in the host (D_{ij}) referred to coordinate axes that coincide with the principal axes of the elliptical cross-section of the inclusion (Fig. 1), in which the solution is developed, are given by:

$$\begin{aligned} D_{xx} &= \beta_{11}\sigma_{xx} + \beta_{12}\sigma_{yy} + \beta_{16}\sigma_{xy} \\ D_{yy} &= \beta_{12}\sigma_{xx} + \beta_{22}\sigma_{yy} + \beta_{26}\sigma_{xy} \\ 2D_{xy} &= \beta_{16}\sigma_{xx} + \beta_{26}\sigma_{yy} + \beta_{66}\sigma_{xy} \end{aligned} \quad (2)$$

where:

$$\begin{aligned} D_{xx} &= \frac{\partial v_x}{\partial x} \\ D_{yy} &= \frac{\partial v_y}{\partial y} \\ D_{xy} &= \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \\ \omega &= \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \end{aligned} \quad (3)$$

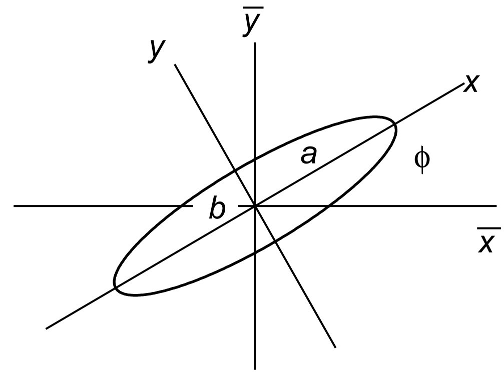

Fig. 1. Elliptical cross-section of cylindrical inclusion and coordinate systems with respect to inclusion shape and viscous anisotropy of the host material.

where v_x and v_y are the components of velocity, and ω is the vorticity. Incompressibility requires that the sum of the normal components of the rate of deformation vanish, giving:

$$\begin{aligned} \beta_{22} &= -\beta_{12} = \beta_{11} \\ \beta_{26} &= -\beta_{16} \end{aligned} \quad (4)$$

The β_{ij} may be expressed in terms of the principal viscosities of the host material and the angle that the principal axes of the inclusion makes with the reference principal axes of anisotropy $\bar{x}, \bar{y}$ in the host (Fig. 1). We may think of the host as a gneiss or schist, with a sub-planar foliation. The viscosity in foliation-parallel shortening or extension is denoted η_n, while that in foliation-parallel shear is denoted η_s. The ratio $m = \eta_n/\eta_s \geq 1$ specifies the strength of anisotropy; Treagus (2002) terms m the anisotropy factor (her δ). If ϕ is the angle between the foliation and the orientation of the inclusion axis (Fig. 1):

$$\begin{aligned} \beta_{11} &= \frac{[(m+1) - (m-1)\cos 4\phi]}{8\eta_n} \\ \beta_{66} &= \frac{[(m+1) + (m-1)\cos 4\phi]}{2\eta_n} \\ \beta_{16} &= \frac{(m-1)\sin 4\phi}{4\eta_n} \end{aligned} \quad (5)$$

The solution derived by Lekhnitskii (1963), and used in Fletcher (2004), is for the additional inhomogeneous stress and velocity fields that mediate between the homogeneous inclusion states of stress, rate of deformation, and vorticity and their equivalents in the far-field. Consequently, the far-field rate of deformation components, indexed with "o", are:

$$\begin{aligned} D_{xx}^o &= 2\beta_{11}s_{xx}^o + \beta_{16}s_{xy}^o \\ D_{yy}^o &= -2\beta_{11}s_{xx}^o - \beta_{16}s_{xy}^o \\ D_{xy}^o &= \beta_{16}s_{xx}^o + \frac{\beta_{66}}{2}s_{xy}^o \end{aligned} \quad (6)$$

where:

$$\begin{aligned} s_{xx} &= \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \\ s_{xy} &= \sigma_{xy} \end{aligned} \quad (7)$$

where σ_{xx}, σ_{yy}, and σ_{xy} are the components of stress. The inclusion rate of deformation components, indexed with *, are:

$$\begin{aligned} D_{xx}^* &= 2\beta_{11}^*s_{xx}^* + \beta_{16}^*s_{xy}^* \\ D_{yy}^* &= -2\beta_{11}^*s_{xx}^* - \beta_{16}^*s_{xy}^* \\ D_{xy}^* &= \beta_{16}^*s_{xx}^* + \frac{\beta_{66}^*}{2}s_{xy}^* \end{aligned} \quad (8)$$

The solution for the stress components and vorticity in an anisotropic viscous inclusion, with constitutive relations (8) embedded in a homogenous matrix with relations (6) is obtained from the algebraic equations (A7) in the appendix of Fletcher (2004), but these contain several typographical errors. Further, a marked condensation of these equations may be made, using relations I later found for an incompressible medium:

$$\begin{aligned}(\alpha_1+\alpha_2)\beta_{11}-\beta_{16} &= 0\\ (\beta_1+\beta_2)\beta_{11} &= \tfrac{\sqrt{m}}{2\eta_n}\end{aligned} \tag{9}$$

where the quantities $\alpha_1, \alpha_2, \beta_1, \beta_2$ are the real and imaginary parts of the roots of the characteristic equation for the 4th-order partial differential equation in the stress function for the case of an anisotropic medium, equivalent in the isotropic case to the biharmonic equation (Lekhnitskii, 1963, p. 136). The orientation of the principal axes of anisotropy with respect to the inclusion axes in either host or inclusion may be arbitrary. With the simplification afforded by (9), the velocity boundary conditions (Fletcher, 2004, equation A7) then become:

$$\begin{aligned}\sigma^*-\sigma^o &= \tfrac{(a^2-b^2)}{(a^2+b^2)}\left(s^*_{xx}-s^o_{xx}\right)\\ \omega^* &= \omega^o+\left(\frac{a}{b}-\frac{b}{a}\right)\tfrac{m}{4\eta_n}\left(s^*_{xy}-s^o_{xy}\right)\\ \left[\tfrac{m}{\eta_n}+4\nu\beta^*_{11}\right]s^*_{xx}+2\nu\beta^*_{16}s^*_{xy} &= \left[\tfrac{m}{\eta_n}+4\nu\beta_{11}\right]s^o_{xx}+2\nu\beta_{16}s^o_{xy}\\ 2\beta^*_{16}s^*_{xx}+\left(\nu\tfrac{m}{\eta_n}+\beta^*_{66}\right)s^*_{xy} &= 2\beta_{16}s^o_{xx}+\left(\nu\tfrac{m}{\eta_n}+\beta_{66}\right)s^o_{xy}\end{aligned} \tag{10}$$

where:

$$\begin{aligned}\nu &= \tfrac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)\\ \sigma &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})\end{aligned} \tag{11}$$

The far-field rate of deformation components may be substituted for the stress components, using (6). Specification of the current state of flow is general, since arbitrary values of rate of deformation components, $\overline{D}_{xx}, \overline{D}_{xy}$ referred to the principal axes of anisotropy, may be specified, and:

$$\begin{aligned}D^o_{xx} &= \overline{D}_{xx}\cos 2\phi+\overline{D}_{xy}\sin 2\phi\\ D^o_{xy} &= -\overline{D}_{xx}\sin 2\phi+\overline{D}_{xy}\cos 2\phi\\ s^o_{xx} &= \overline{s}_{xx}\cos 2\phi+\overline{s}_{xy}\sin 2\phi = 2\eta_n\left(\overline{D}_{xx}\cos 2\phi+\tfrac{\overline{D}_{xy}}{m}\sin 2\phi\right)\\ s^o_{xy} &= -\overline{s}_{xx}\sin 2\phi+\overline{s}_{xy}\cos 2\phi = 2\eta_n\left(-\overline{D}_{xx}\sin 2\phi+\tfrac{\overline{D}_{xy}}{m}\cos 2\phi\right)\end{aligned} \tag{12}$$

For a deformable, isotropic inclusion of viscosity η^*:

$$\begin{aligned}\beta^*_{11} &= \tfrac{1}{4\eta^*}\\ \beta^*_{16} &= 0\\ \beta^*_{66} &= \tfrac{1}{\eta^*}\end{aligned} \tag{13}$$

Solving (10) for this case, we obtain the inclusion vorticity

$$\begin{aligned}\omega^* = \omega^o+\frac{1}{2}\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{\sqrt{m}R}{R\nu\sqrt{m}+1}\right)\Big[&\left(-\overline{D}_{xx}\sin 2\phi+\overline{D}_{xy}\cos 2\phi\right)\\ &-\frac{1}{R}\left(-\overline{D}_{xx}\sin 2\phi+\frac{\overline{D}_{xy}}{m}\cos 2\phi\right)\Big]\end{aligned} \tag{14}$$

where $R = \eta^*/\eta_n$, and the rate of deformation components:

$$\begin{aligned}D^*_{xx} &= \frac{\nu}{(R\sqrt{m}+\nu)}\Big[\left(\overline{D}_{xx}\cos 2\phi+\overline{D}_{xy}\sin 2\phi\right)\\ &\quad+\frac{\sqrt{m}}{\nu}\left(\overline{D}_{xx}\cos 2\phi+\frac{\overline{D}_{xy}}{m}\sin 2\phi\right)\Big]\\ D^*_{xy} &= \frac{1}{(R\nu\sqrt{m}+1)}\Big[\left(-\overline{D}_{xx}\sin 2\phi+\overline{D}_{xy}\cos 2\phi\right)\\ &\quad+\nu\sqrt{m}\left(-\overline{D}_{xx}\cos 2\phi+\frac{\overline{D}_{xy}}{m}\cos 2\phi\right)\Big]\end{aligned} \tag{15}$$

3. Finite deformation

While numerical modeling of flow in an anisotropic incompressible viscous fluid has been done (e.g., Kocher and Mancktelow, 2006; Pettit et al., 2007), consideration of finite inclusion deformation in such a fluid is useful geologically. During the continued deformation of an inclusion embedded in an anisotropic medium, internal structure develops by deformation of the matrix "foliation", and the host becomes inhomogeneous; however, homogeneity is imposed as a useful approximation.

Further, the homogeneous host solution is appropriate when estimating the bulk behavior of a dispersion of rigid or deformable clasts embedded in an isotropic matrix by self-consistent averaging (Treagus and Treagus, 2001; Treagus, 2002; Fletcher, 2004). The bulk anisotropy is associated with clast alignment and aspect ratio, and need not be predominantly due to an intrinsic anisotropy of the matrix. Inhomogeneous deformation in the host, to which is attributed the bulk properties of the composite material, does not then affect its homogeneity. Mathematical treatment of the finite deformation of elliptical cylindrical inclusions has been extensively studied (e.g., Bilby and Kolbuszewski, 1977; Mulchrone and Walsh, 2006; Mulchrone, 2007). Here, I present simple relations that facilitate computation, and provide a few illustrations.

Consider an elliptical inclusion with semi-axes of length a parallel to the x-axis and b parallel to the y-axis. Let the positive x-axis lie at an angle ϕ to a reference axis, $\overline{x}$, where this may be taken parallel to bedding or to a shear zone (Fig. 1). Considering the change in principal axes and orientation of this ellipse after an infinitesimal time dt, we obtain:

$$\begin{aligned}\frac{1}{a}\frac{da}{dt} &= D_{xx}{}^*\\ \frac{1}{b}\frac{db}{dt} &= D^*_{yy}\\ \frac{d\phi}{dt} &= \omega^*+\left(\frac{a^2+b^2}{a^2-b^2}\right)D^*_{xy}\end{aligned} \tag{16}$$

The procedure used to obtain this result is to update the components of the displacement gradient tensor (F_{ij}), referred to axes $\overline{x}, \overline{y}$, and assigned current values $\overline{F}_{xx}(0) = \overline{F}_{yy}(0) = 1, \overline{F}_{xy}(0) = \overline{F}_{yx}(0) = 0$ using the components ($V_{ij} = \partial v_i/\partial x_j$) of the velocity gradient tensor obtained from (14) and (15). The new, infinitesimally different orientation and aspect ratio of the inclusion are then found. The quantities $D^*_{xx}, D^*_{xy}, \omega^*$ in (16) are given by (14) and (15), with $D^*_{yy} = -D^*_{xx}$. Relations (16) are general for a homogeneous deformation, however these quantities are determined. As Bilby and Kolbuszewski (1977) note, the homogeneous elliptical inclusion might have another rheological behavior – e.g., nonlinear power-law behavior – than the viscous behavior considered here and a solution for such a case can generally be found because the stress, rate of deformation and vorticity within the inclusion will remain homogeneous. Bilby and Kolbuszewski (1977) and Mulchrone and Walsh (2006), among others, have established relationships equivalent to (16), and the former authors give a more general result for an ellipsoid.

4. Rotation of a rigid inclusion

For the rigid inclusion, (14) reduces to:

$$\omega^* = \omega^o + \left(\frac{a^2-b^2}{a^2+b^2}\right)\left(-\overline{D}_{xx}\sin 2\phi + \overline{D}_{xy}\cos 2\phi\right) \quad (17)$$

where the term in the second parentheses is ½ the engineering rate of shear on the plane parallel to the long axis of the inclusion. This result is also precisely that for the isotropic case. An independent check on (17) has been obtained through both a numerical solution and an analytical solution for the incompressible case obtained by a different method (M. Dabrowski, personal communication, 2007). In view of the variety of results already presented for the rotation of the rigid elliptical inclusion in an isotropic viscous medium (e.g., Ghosh and Ramberg, 1976), none are given here.

5. Inviscid and incompressible inclusion

The case of an inviscid and incompressible inclusion provides, in the limit of infinite aspect ratio, a solution for a "flanking structure" (Passchier, 2001; Grasemann and Stuwe, 2001; Grasemann et al., 2003; Exner et al., 2004; Kocher and Mancktelow, 2005, 2006; Mulchrone, 2007). Flanking structures might also be modeled by choosing a finite viscosity for the inclusion, or, in view of the possibility of choosing arbitrary rheological behavior for the inclusion, treat the case in which slip requires a fixed ratio of shear stress to normal stress (Marcin Dabrowski, personal communication, 2007).

If the velocity matching conditions are written in a form obtained prior to the algebraic reduction that yields the general set (10), the vanishing of the deviatoric stress in an inviscid inclusion then results in simpler relations. These reduce to two pairs of equations. One pair takes the form:

$$\begin{aligned}\frac{b}{a}\beta_{11}(\beta_1+\beta_2)\sigma^*_{xx} + D^*_{xx} &= \frac{b}{a}\beta_{11}(\beta_1+\beta_2)\sigma^o_{xx} + 2\beta_{11}s^o_{xx} + \beta_{16}s^o_{xy}\\ \frac{a}{b}\beta_{11}(\beta_1+\beta_2)\sigma^*_{yy} + D^*_{yy} &= \frac{a}{b}\beta_{11}(\beta_1+\beta_2)\sigma^o_{yy} - 2\beta_{11}s^o_{xx} - \beta_{16}s^o_{xy}\end{aligned} \quad (18)$$

Here, we do not expand the inclusion rates of deformation using (8). Instead, we note that the deviatoric stress in it vanishes, so that:

$$\sigma^*_{xx} = \sigma^*_{yy} = \sigma^* \quad (19)$$

Further, incompressibility requires that the normal components of the rate of deformation be equal in magnitude and opposite in sign. Thus, after substitution, (18) immediately yields:

$$\begin{aligned}D^*_{xx} &= \left(\frac{\sqrt{m}}{\nu}\right)\left(\overline{D}_{xx}\cos 2\phi + \frac{\overline{D}_{xy}}{m}\sin 2\phi\right) + \left(\overline{D}_{xx}\cos 2\phi + \overline{D}_{xy}\sin 2\phi\right)\\ \sigma^* &= \sigma^o - \left(\frac{a^2-b^2}{a^2+b^2}\right)s^o_{xx}\end{aligned} \quad (20)$$

From the remaining two velocity conditions we obtain, in like manner:

$$\begin{aligned}D^*_{xy} &= (\nu\sqrt{m})\left(-\overline{D}_{xx}\sin 2\phi + \frac{\overline{D}_{xy}}{m}\cos 2\phi\right) + \left(-\overline{D}_{xx}\sin 2\phi + \overline{D}_{xy}\cos 2\phi\right)\\ \omega^* &= \omega^o + \sqrt{m}\left(\frac{b}{a} - \frac{a}{b}\right)\left(-\overline{D}_{xx}\sin 2\phi + \frac{\overline{D}_{xy}}{m}\cos 2\phi\right)\end{aligned} \quad (21)$$

In the relations for the rates of deformation, the second terms are the far-field values in inclusion coordinates.

An expression for the relative motion of the two sides, of a thin inviscid elliptical inclusion may be derived. Given that the velocity field within the inclusion is:

$$\begin{aligned}v^*_x &= D^*_{xx}x + \left(D^*_{xy} - \omega^*\right)y\\ v^*_y &= \left(D^*_{xy} + \omega^*\right)x + D^*_{yy}y\end{aligned} \quad (22)$$

Then, the relative motion is:

$$\Delta v^*_x = v^*_x(0,b) - v^*_x(0,-b) = 2\left(D^*_{xy} - \omega^*\right)b \quad (23)$$

which is, from (21):

$$\begin{aligned}\Delta v^*_x &= 2a\sqrt{m}\left(-\overline{D}_{xx}\sin 2\phi + \frac{\overline{D}_{xy}}{m}\cos 2\phi\right)\\ &\quad + 2b\left[\left(-\overline{D}_{xx}\sin 2\phi + \overline{D}_{xy}\cos 2\phi\right) - \omega^o\right]\\ &\cong 2a\sqrt{m}\left(-\overline{D}_{xx}\sin 2\phi + \frac{\overline{D}_{xy}}{m}\cos 2\phi\right), \frac{a}{b} \gg 1\end{aligned} \quad (24)$$

If this quantity is positive, the slip is dextral, if it is negative, slip is sinistral. The term in parentheses in the 2nd line of (24) is proportional to the far-field shear stress in the medium.

6. Deformable isotropic inclusion in an anisotropic medium in finite foliation-parallel shear

An example that is likely to find application in the interpretation of natural structures is that of an isotropic inclusion embedded in a medium undergoing homogeneous simple shear in the plane of its foliation. For foliation oblique to the shear plane, the deformation is more complex, and is described by the evolution of three quantities, the axial ratio, a/b, the orientation of the long axis to the shear plane, ϕ, and the angle of the foliation to the shear plane, which is here taken to be fixed at zero. Numerical integration of (16) provides the trajectories of a, or a/b, and ϕ with time. Alternatively, we may consider the trajectory of a point in ϕ, a/b – phase space as done by Bilby and Kolbuszewski (1977) and Mulchrone and Walsh (2005, Fig. 8). An example of a phase-space plot is given in Fig. 2.

Here, the behavior of the inclusion will generally depend on three viscosities, that of the isotropic inclusion, η^*, and the two principal viscosities of the host, η_n and η_s. When both host and inclusion are isotropic viscous fluids, with $m = 1$, the rheological behavior depends on the single ratio, $R = \eta^*/\eta$, where η is the isotropic host viscosity. Three classes of behavior are identified as functions of R (Bilby and Kolbuszewski, 1977; Mulchrone and Walsh, 2006). (i) When $R < 2$, inclusions with any initial values of orientation to the shear plane ϕ, and aspect ratio a/b, will, with sufficient shear strain, have their long axes approach the shear plane and their aspect ratios $\rightarrow \infty$. (ii) When $2 < R < R_1$, a stationary point is present in ϕ, a/b – phase space at $\phi = 0$ and $(a/b)_C = R/(R-2)$. An inclusion with these initial parameters will retain them in the shearing flow. Trajectories giving the evolution of inclusion orientation and aspect ratio in phase space will be closed loops within some region around this stationary point, and any inclusion whose initial values lie in this region will undergo periodic oscillatory motion in its aspect ratio and long axis orientation. Outside of this region, inclusions are again drawn out without limit and have their long axes approach the shear plane. The lower boundary between these two classes of behavior is given by the condition that $(a/b)_C$ just achieve a positive, if infinite, value – i.e., $R = 2$. (iii) When $R > R_1 \cong 3.40$ (see Bilby and Kolbuszewski,

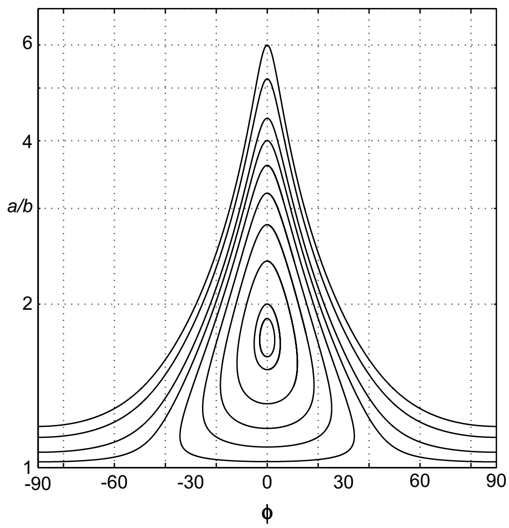

Fig. 2. Representative trajectories of an isotropic elliptical inclusion in ϕ, a/b – phase space. See text for further discussion.

1977), all initial values of ϕ and a/b give rise to periodic motions, which are either oscillatory, for initial values near the stationary point, or rotational otherwise (see Fig. 2).

For the isotropic inclusion in an anisotropic host, $m>1$, the limiting values of R that define the three classes of behavior are different from those in the isotropic host case, with a further dependence on m. We find $(a/b)_C$ from the condition that $\mathrm{d}\phi/\mathrm{d}t=0$ at $\phi=0$. Using the results given, we obtain:

$$\left(\frac{a}{b}\right)_C = \frac{\sqrt{m}\left[1+\sqrt{1+R(mR-2)}\,\right]}{mR-2} \tag{25}$$

Since $mR=\eta^*/\eta_s$, the condition for the boundary between classes (i) and (ii) in simple shear parallel to the foliation is independent of the viscosity in foliation-parallel extension or shortening, η_n. It is just $mR=2$. Thus, even for $R = \eta^*/\eta_n < 1$, inclusions may undergo periodic motions in ϕ, a/b – space. Here, it is the viscosity in shear that is pertinent. The result (25) is reasonably well-approximated, at least for m not too large, by the empirical fit:

$$\left(\frac{a}{b}\right)_C \cong \frac{m^{3/4}R}{m^{3/4}R-2} \tag{26}$$

where $m^{3/4}R = \eta^*/(\eta_n\eta_s^3)^{1/4}$. This relationship indicates that the dependence on η_n is relatively small. Of course, the proper condition on the position of the boundary between (i) and (ii) has no dependence on η_n.

The boundary between classes of behavior (ii) and (iii) was obtained empirically, by judging between initial inclusion trajectories that showed periodicity and those that did not, for a fixed value of m and increasing R. Using values of R_1 obtained for1 - $\le m \le 10$, and requiring that $R_1(1)=3.40$, we obtained the fit:

$$m^N R_1 \cong 3.40 \qquad N = 0.786 \cong 0.8 \tag{27}$$

This result again indicates a relative insensitivity to η_n.

In pure shear parallel to the foliation, we might well expect a reversal of roles, with η_n dominant, and η_s playing only a modest role in the behavior. This intuition can be verified by using the relations defining the rates of change in aspect ratio and orientation with $\overline{D}_{xx} \ne 0, \overline{D}_{xy} = 0$. Evidently, a more complicated behavior will be manifested in "transpression" or "transtension," or in the more general histories of non-constant far-field deformation – i.e., local bulk deformation – that generally apply in natural deformations.

Fig. 2 shows several trajectories for the case $m=3$, $R=2$. In positive, or dextral-, shear, all points describing inclusion behavior move in a counterclockwise sense, on the trajectories. The position of the stationary point is within the smallest trajectory on the $\phi=0$ axis. Closed trajectories surrounding this point correspond to periodic, oscillating behavior. The long axis swings between its maximum and minimum values, but is never outside of the range from $+45^\circ$ and -45° from the shear plane. The non-closed trajectories are periodic, with full rotation.

7. Conclusions

A solution for the deformation of an anisotropic viscous inclusion in a homogeneous incompressible anisotropic viscous fluid, with welded host/inclusion interface has been presented. In application to rigid inclusions, it is shown that the rate of rotation is independent of the anisotropy and identical to that for a rigid inclusion in an isotropic viscous medium. A special solution is derived for an inviscid incompressible inclusion and it is applied to determine the sense and rate of "slip" across a thin, very weak inclusion, such as is implicated in the formation of flanking structures. Results obtained for finite deformation in simple shear with an isotropic host are extended to the case of an anisotropic viscous host to provide an approximation to the behavior in which the deformation disrupts its homogeneity.

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